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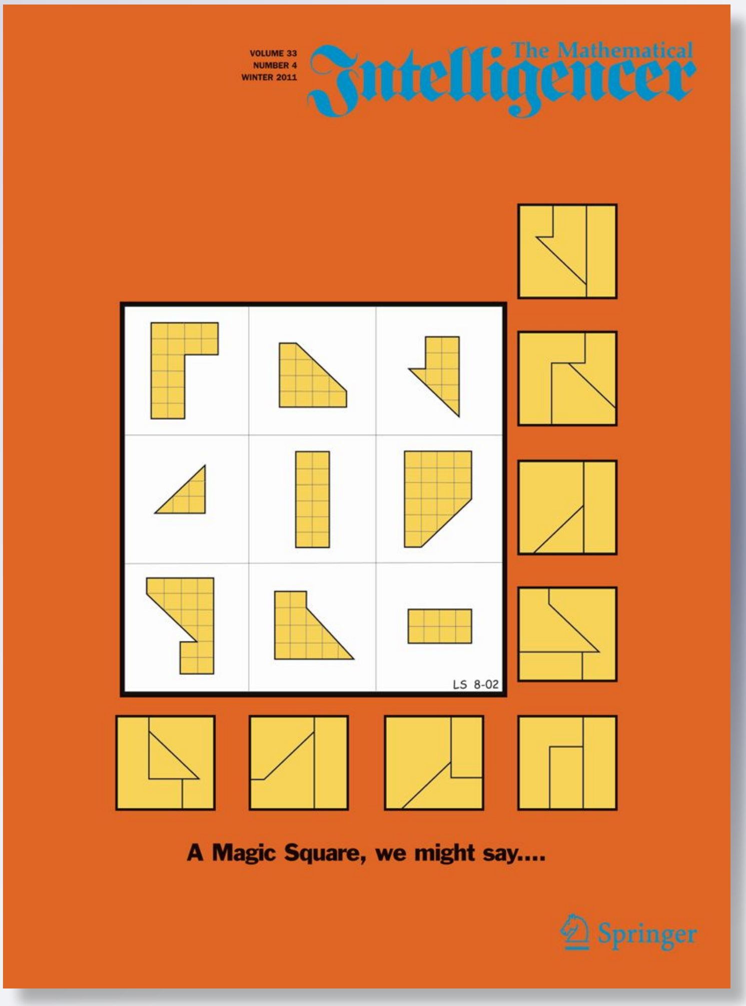

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Mysterious Movable Models

JEAN PEDERSEN AND TIBOR TARNAI

This article is about models that move in surprising ways. The idea of duality permeates and motivates much of the thinking related to this topic. We will describe two distinct types of bar-and-joint models. The first type involves pairs of pyramids (which are self-dual), whereas the second type involves two different, but dual, polyhedra. In the first type we have two separate, but identical, models with the property that when one is "put inside" the other, it results in a movable configuration, although during the movement neither of the pyramids themselves changes shape. In the second type, all of the edges are connected to each other in such a way that, even in theory, they cannot be separated into two parts, and the entire configuration moves in such a way that sets of vertices become closer or farther away from the center of the model. We call the first type of the bar-and-joint models *separable*, and the second type we call *connected*.

1982–1987: How It All Started

In the winter of 1987, at the invitation of Jean Pedersen (JP), Tibor Tarnai (TT) visited Santa Clara University and presented a colloquium talk about a pair of separable tetrahedra that had been shown to him in 1982 by L. Tompos, Jr., who was then a second-year undergraduate student of the Hungarian Academy of Craft and Design (Fig. 1). The structure consists of the bar-and-joint frames of two identical regular tetrahedra, one fitted inside the other. Note that the six edges (bars) of the inner tetrahedron are in contact with, and at right angles to, the six edges of the outer tetrahedron.[1] This model is essentially the same as the one in Fuller's book [2].

These six contact points constitute six constraints of degree one, which in general are sufficient to prevent three translations and three rotations (relative motions between two rigid bodies) in three-dimensional space. Therefore we would expect the structure of Tompos to be rigid.

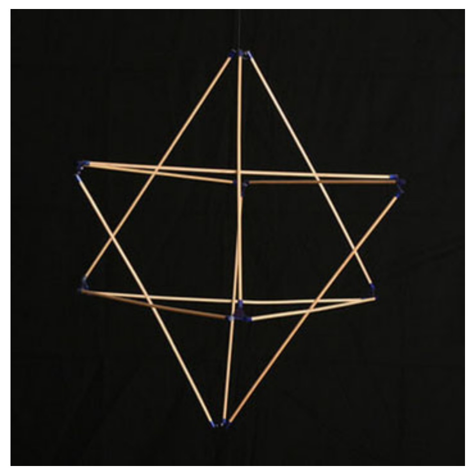

Figure 1 (All photos by Chris Pedersen)

Surprisingly, when we hold one tetrahedron of the physical model in our hands, we discover that we may easily move the other one in almost any direction with the crossing edges sliding over each other.

This revelation came as a pleasant surprise to JP, since she had recently been given what we will call a connected bar-and-joint model, shown in Figure 2. This remarkable model is constructed from 24 bars connected by flexible joints at 8 vertices of degree 3 and at 6 vertices of degree 4. When the 3-degree vertices are outermost (Fig. 2a), the innermost 4-degree vertices are the vertices of a phantom octahedron; when the 4-degree vertices are outermost (Fig. 2b), the innermost 3-degree vertices are the vertices of a phantom cube. The size of the phantom cube, and of the phantom octahedron, vary as vertices of the same degree are moved closer or farther away from the center of the model. There is

[1] A physical model of this structure can be built from Geo-D-Stix bars and joints. The models shown in the photographs of this article were constructed from wooden sticks and plastic movable connectors obtained from Avionics Plastics, which is now out of business.

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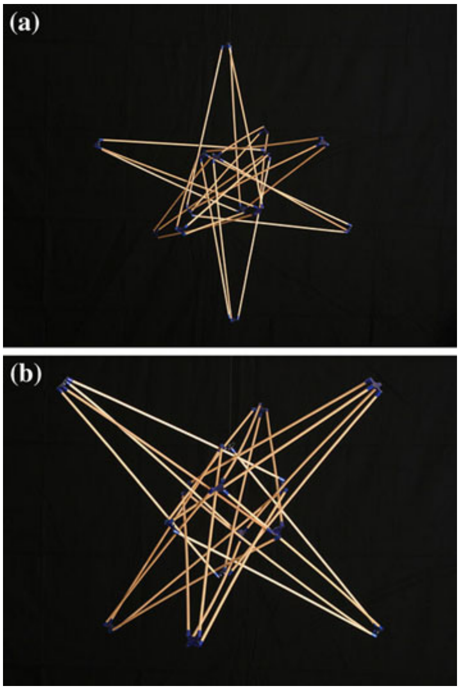

Figure 2

Name (of Platonic Solid)	V (# of vertices)	E (# of edges)	F (# of faces)
Tetrahedron	4	6	4
Cube (**H**exahedron)	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Figure 3

an intermediate position where the 14 vertices would be the vertices of a phantom rhombic dodecahedron.

At that time, JP had conjectured that the bar-and-joint model of Figure 2 had something to do with the cube and the octahedron being *duals* to each other, in the sense that they both have the same number of edges, the number of faces for either one is the number of vertices for the other (Fig. 3).

Let us refer to the model of Figure 2 as the connected **H-O** model. JP asked herself: Can one construct an analogous connected **D-I** model?

With some experimentation, JP eventually managed to build such a model (Fig. 4). This bar-and-joint model doesn't flex as far, nor as easily, as the **H-O** model; but the miracle is that it does flex. JP believes that perhaps the **D-I** model is restricted in its motion just by the nonzero thickness of the bars. (Volunteers are sought to try to build a more flexible **D-I** model by using thinner, or longer, bars!)

Observe that in both the connected **H-O** and connected **D-I** models, one of the original dual polyhedra models is not rigid. Namely, in the **H-O** model, the bar-and-joint **H** is not rigid, and in the **D-I** model, the bar-and-joint **D** is not rigid. Another characteristic shared by these two models is that the only available motion is that of shrinking/expanding the phantom polyhedron. Thus the combined bar-and-joint arrangement for the **H-O** and the **D-I** produces a kind of rigidity in **H** and **D** that was not present in the parent models from which they came.

January 1987: Surprising Discovery

After TT's talk, JP reflected on the fact that the regular tetrahedron is self-dual. But the tetrahedron is just one of an infinite class of self-dual pyramids (Fig. 5). Would it perhaps be possible to construct other separable models analogous to the movable pair of tetrahedra, but with pyramids having a base with more than 3 sides?

Let us call a pyramid with an ***n*-*sided*** base a ***n*-*pyd***, and call the model of Figure 1 a separable **3-*pyd***^{2}, because it consists of two **3-*pyds***. So, how would one build a separable **4-*pyd***^{2}?

First build two bar-and-joint **4-*pyds*** with all bars the same length.[2] Each of these will be, by itself, unstable, for the base can be deformed into a nonregular quadrilateral. Next, place one **4-*pyd*** inside the other with their apexes pointing in opposite directions ("north and south poles") and their bases in a convex position about the "equator." This completes the

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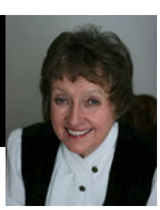


JEAN PEDERSEN teaches calculus courses as well as problem-solving and combinatorics. Her research interests include polyhedral geometry, elementary number theory, and combinatorics. She also serves as an Associate Editor of the Mathematical Association of America's Spectrum Series.

Department of Mathematics
and Computer Science
Santa Clara University
Santa Clara, CA 95053
USA
e-mail: jpedersen@scu.edu

TIBOR TARNAI received degrees in civil engineering and applied mathematics in Budapest. His special field is structural mechanics. More generally, he is interested in mechanical and geometrical problems, and in their interactions, such as packing, covering, and rigidity problems

Department of Structural Mechanics
Budapest University of Technology
and Economics
Budapest H-1521
Hungary
e-mail: tarnai@ep-mech.me.bme.hu

[2]All the ***n*-*pyds*** shown in this article have bars of the same length. Makai and Tarnai [6] briefly mention ***n*-*pyds*** for $3 \leq n \leq 7$ where the length of the lateral edges and that of the base edges are different.

Figure 4

n		V	E	F
3		4	6	4
4		5	8	5
5		6	10	6
⋮		⋮	⋮	⋮
n		$n+1$	$2n$	$n+1$

Figure 5

separable ***4-pyd***^{2}. It is at this point that a surprising thing happens: *Both pyramids then BECOME RIGID*. Furthermore, if one of the pyramids is held fixed, the other can be rotated about the axis joining the north and south poles. One can construct similar bar-and-joint separable ***n-pyd***^{2} for any $n \geq 4$.[3]

Spring 2007: Known Technical Details

Between 1987 and 2007, both authors showed these models to students and friends, without being able to present a complete mathematical explanation of their unintuitive behavior.

TT and many of his colleagues have conducted research to discover the properties, and variations, of Tompos's pair of tetrahedra [7, 8, 9, 10] , along with a pair of tetrahedra where the edges of the tetrahedra are face diagonals of a rectangular parallelepiped or a general parallelepiped (also called a rhombohedron) [6]; and they also conducted research to investigate their applicability in practice [1, 5].

For Tompos's separable pair of tetrahedra, the following mathematical problem was analyzed [9]. Consider a cube in 3-space. Draw all the diagonals of all its faces. These will constitute the edges of two regular tetrahedra, both inscribed into the cube. Fix one of these tetrahedra, and try to move the other one with the restriction that each pair of edges of both tetrahedra, which were originally diagonals of the same face of the cube, should still remain coplanar (i.e., intersect, be parallel, or coincide). The question is whether such motions are possible.

Looking for such motions in the form $\Phi(x) = Ax + b$, where A is a 3×3 matrix with determinant $+1$ representing a rotation about an axis through an angle and b is a vector in 3-space representing a translation, they established that the free motions of the tetrahedra constitute 1- and 2-dimensional submanifolds of the 6-dimensional manifold of all unconstrained motions. The submanifolds intersect along a line or at a point where the infinitesimal degree of freedom of the motion increases to 3, although no 3-degree-of-freedom finite motions exist [10]. Here there is a bifurcation of the motions. This happens to the tetrahedra, for instance, in the basic position (Fig. 1), where the convex hull of the two tetrahedra is a cube.

So far, kinematic analysis of the motions of the connected models has been fragmentary. Some of our new results concerning their basic properties are briefly reported in the following text.

The connected **T-T** model has the same number of bars, the same number of crossings of bars, and the same number of joints (vertices) as the separable pair of tetrahedra of Tompos has, but it is much more floppy. In a position possessing tetrahedral symmetry, it has two infinitesimal degrees of freedom and additionally two finite degrees of freedom that are preserved in any position. This is why the model loses tetrahedral symmetry so easily when handled.

The connected **H-O** model contains 24 bars, which cross each other at 36 points, and 14 joints. For each bar and each crossing, a constraint equation can be set up, in which the coordinates of the joints are the unknowns. The constraint equation for a bar expresses the fact that the distance between the endpoints of the bar is equal to the length of the bar. The constraint equation for a crossing expresses the fact that the endpoints of two crossing bars are coplanar, or in other words, the volume of the tetrahedron spanned by the endpoints of the two bars is zero. The rank of the 60×42 Jacobian matrix of the constraint functions is 35. That means that the model has 7 degrees of freedom. If we remove the 6 degrees of freedom of the rigid motion (motion of the entire model in 3-space), we still have one (at least infinitesimal) degree of freedom. In fact, this one degree of freedom is a finite degree of freedom. That means that, despite being highly overconstrained, the model is able to move with one degree of freedom in such a way that in each position the model has octahedral symmetry.

Bar crossings, however, are unilateral constraints. Consequently it can happen that two bars theoretically crossing each other are physically not in contact. Then crossing constraint does not work any longer. In the extreme case where all bars are crooked, and the respective bars do not touch each other, the model moves with 12 degrees of freedom. But if all constraints are maintained, then the model has only a one-degree-of-freedom motion. If a certain number of contacts are lost and a new degree of freedom appears, the model may lose octahedral symmetry and move into a shape of lower symmetry.

[3]Pictures of some other separable ***n-pyd***^{2} models, along with the **H-O** and **D-I** models, can be found at http://www.pqphotography.com/Mathematical-Models/MMM.

The connected **D-I** model contains 60 bars, which cross each other at 240 points, and 32 joints. The rank of the 300×96 Jacobian matrix is 89. Again, we have 7 degrees of freedom, 6 of which are the trivial degrees of freedom of the rigid motion of the model. So it turns out that, if all constraints are maintained, the model has a one-degree-of-freedom finite free motion where, in each position, the model retains icosahedral symmetry.

In this survey we can not go further with the details of these extraordinary models. In the bibliography, brief annotations give a guide to the contents of the research articles.

2012... : Challenges for the Reader

Since we haven't exhausted all the possible dual pairs of polyhedra, there remain many open questions. Here are just a few:

1) Do *any* other separable bar-and-joint models exist?
2) Recall that the regular tetrahedron is a special case of the infinite set of self-dual n-pyds. Analogously, Figure 6 displays dipyramids (with a regular n-gon at each equator) and prisms (with a regular n-gon for each base). Notice that when $n = 4$, the dual polyhedra are our old friends **H** and **O**. It is natural to ask: Is it possible to construct bar-and-joint models analogous to the **H-O** bar-and-joint model of Figure 2 (where $n = 4$) for pairs of polyhedra shown in Figure 6 when $n \neq 4$? Howard [4] implies that for $n \geq 5$ these make splendid articulating polyhedra, but he also observes that for $n = 3$ his method of constructions "produces a rather loose articulating model." Why loosely articulated?
3) The duals of the Archimedean solids (vertex-congruent semiregular convex polyhedra) are sometimes called the Catalan polyhedra. Is it possible to construct an analogous connected bar-and-joint model for an Archimedean-Catalan pair of polyhedra?
4) Thus far we have considered only polyhedra whose symmetry groups are tetrahedral, octahedral, icosahedral, cyclic, or dihedral. But every convex polyhedron has its dual polyhedron, so this seems quite restrictive. We venture to ask, "Can a flexible bar-and-joint model be constructed from a pair of dual polyhedra that does not belong to any of the above-mentioned symmetry types?" Our guess is that the answer to this is "no," but we would love to be surprised.

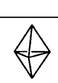
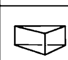
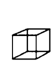
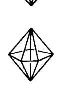
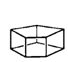
n		V	E	F	n		V	E	F
3		5	9	6	3		6	9	5
4		6	12	8	4		8	12	6
5		7	15	10	5		10	15	7
⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮
n		$n+2$	$3n$	$2n$	n		$2n$	$3n$	$n+2$

Figure 6

5) Howard's article [4] mentions that "Polyhedral toruses produce interesting articulating models" and gives one example (without mentioning what the dual of that torus would be). A particularly interesting torus-like configuration is the rotating ring of regular tetrahedra (constructions for braiding this model from straight strips of paper can be found in Chapter 6 of [3]). It is natural to ask: "Could a separable, or connected, bar-and-joint model be produced from such a model?"

If any readers can make progress on any of these questions, we certainly hope they will share their knowledge with us.

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[1] Chen, H.-W. 1991, *Kinematics and introduction to dynamics of a movable pair of tetrahedra*. M. Eng. Thesis, Dept. of Mechanical Engineering, McGill University, Montreal, Canada.
This provides a thorough description of the kinematics and dynamics of the pair of tetrahedra applied in a novel robotic manipulator called double tetrabot.

[2] Fuller, R. B. 1975, *Synergetics: Exploration in the geometry of thinking*, p. 7. New York: Macmillan.

[3] Hilton, P., Pedersen, J. 2010, *Mathematical Tapestry: Demonstrating the Beautiful unity of Mathematics*, Cambridge University Press.
Figures 5 and 6 are reproduced, with permission, from this book.

[4] Howard, Ian P. 1978–1979, Articulating models, *J. Recreational Mathematics*, 11:3, 190–195.

[5] Hyder, A., Zsombor-Murray, P. J. 1990, Design, mobility analysis and animation of a double equilateral tetrahedral mechanism. In *Proceedings of International Symposium on Robotics and Manufacturing*, ASME Press Series, Vol. 3, ISSN 1052-4150, pp. 49–56.
This shows how to design a real double tetrahedral mechanism satisfying precisely all geometrical requirements (e.g., exact intersection of corresponding edges) and how to use it as a robotic joint.

[6] Makai, E., Tarnai, T. 2000, Overconstrained sliding mechanisms. In *IUTAM-IASS Symposium on Deployable Structures: Theory and Applications. Proceedings of the IUTAM Symposium held in Cambridge, U.K., 6–9 September 1998* (S. Pellegrino, S. D. Guest, eds.), pp. 261–270. Dordrecht: Kluwer Academic Publishers.
This presents a generalization of the pair of tetrahedra. It investigates motions of bar structures consisting of two congruent tetrahedra, whose edges are defined by face diagonals of a rectangular parallelepiped.

[7] Stachel, H. 1988, Ein bewegliches Tetraederpaar. *Elemente der Mathematik* 43, 65–75.
This provides a geometrical proof pertaining to finite motions of the pair of teterahedral frameworks, and maps typical surfaces traced by an edge midpoint and by a vertex.

[8] Tarnai, T., Makai, E. 1988, Physically inadmissible motions of a movable pair of tetrahedra. In *Proceedings of the Third International Conference on Engineering Graphics and Descriptive Geometry* (S. M. Slaby, H. Stachel, eds.), vol 2, pp. 264–271. Vienna: Technical University.

This paper deals with those motions of the pair of tetrahedra that cannot be realized by a physical model, that is, with motions where the points of intersection of straight lines of some respective edges are not internal points of the edges.

[9] Tarnai, T., Makai, E. 1989a, A movable pair of tetrahedra. *Proceedings of the Royal Society of London* A423, 419–442.

This proves the existence of all finite motions of a pair of tetrahedra.

[10] Tarnai, T. Makai, E. 1989b, Kinematical indeterminacy of a pair of tetrahedral frames. *Acta Technica Acad. Sci. Hung*. 102, 123–145.

This investigates infinitesimal movability of a pair of tetrahedra.