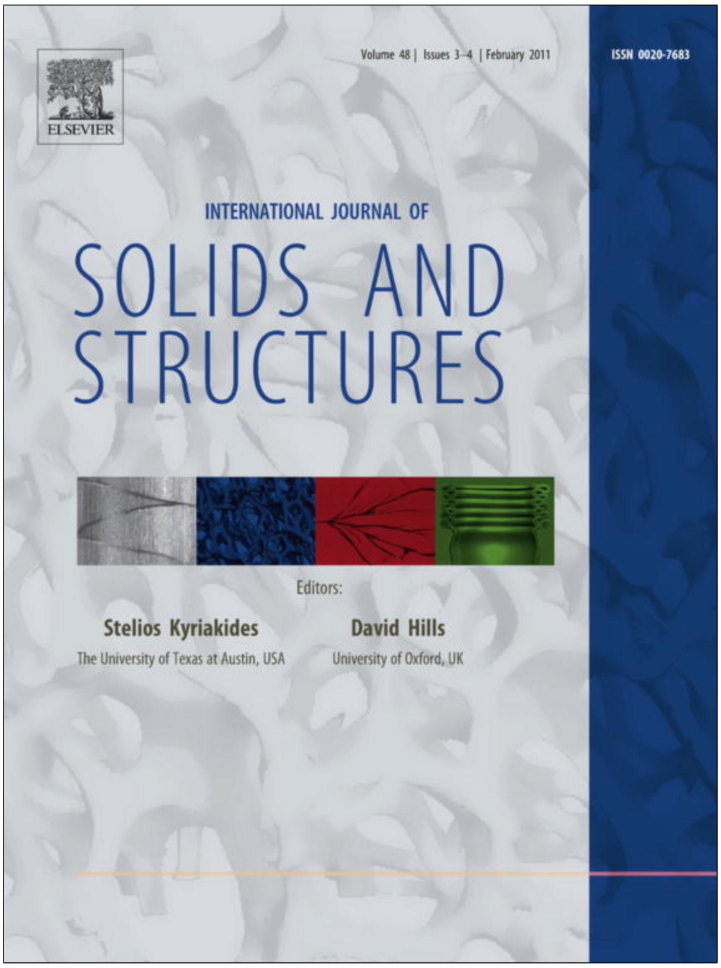
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Extended truss theory with simplex constraints

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ABSTRACT

This paper traces a way of generalization of the classical truss theory: in addition to the kinematic constraint expressing the distance between two nodes connected by a bar element, other similar constraints involving three and four nodes are introduced. Derived from energy principles, a general tangent stiffness formulation is given. Possible mechanical interpretations as well as problems of pre-stressing are also discussed.

1. Introduction

Tangent stiffness formulation of pin-jointed structures has been studied more or less directly in several papers in the recent years. Although there are various approaches to the interpretation of a tangent stiffness matrix, it is common to distinguish three of its components due to (a) *material stiffness* (Guest, 2006), (b) the modifying effect of member forces and (c) *geometrical stiffness* (e.g. Przemieniecki, 1968). However, care must be taken with these denominations, as 'geometrical stiffness', e.g. in Schenk (2006) is applied for the sum of members (b) and (c), while Tarnai and Szabó (2002) use 'complementary stiffness' for the negative of members (b) and (c) together. Guest (2006) refers to members (a) and (b) together as 'modified material stiffness' and calls the matrix corresponding to member (c) the 'stress matrix'. The term *stress* is also of doubtful usage anyway: in the theory of mathematical rigidity (concerned only with problems of prestress stability without respect to material properties, i.e. members (b) and (c) of the tangent stiffness only), 'stress' is applied for a member force over member length (see e.g. Connelly and Whiteley, 1996), while other sources use *tension coefficient* (e.g. Southwell, 1920) or *force density* (e.g. Schek, 1974) with the same meaning.

As was emphasized in Guest (2006), different understanding of the same tangent stiffness is mainly due to different approaches of each field of science, as well as to the way of derivation of the tangent stiffness. The main objective of this paper is to provide a potential energy-based formulation of the tangent stiffness and extended definition of 'stress' for a generalized truss model using the Hellinger–Reissner principle (Washizu, 1982). The general feature of our model comes from the introduction of a generalized kinematic constraint which we call *simplex constraint*. The denomination refers to the physical content of such a constraint, that is, it may express the 'volume' of an arbitrary simplex instead of a simple length (of a bar); for instance, area of a triangle spanned by three nodes in a two- or higher-dimensional space or volume of a tetrahedron in three- (or, theoretically, higher-) dimensional space. Conditions of applicability of simplices with zero volume (called *degenerate simplices* hereinafter) are also analysed.

With the help of this generalized truss model, we want to put different modelling techniques (e.g. used for traditional truss structures or for bar structures with sliding connections) in a unified framework, as well as to suggest some other applications. We will follow a similar path as in Tarnai and Szabó (2002) discussing traditional bar-and-joint assemblies or in Kovács and Tarnai (2009) for the spherical adaptation of the same.

The outline is as follows: in Section 2, a brief resume of the Hellinger–Reissner principle and the corresponding tangent stiffness formulation for bar-and-joint assemblies (trusses) is given. Afterwards, Section 3 provides an extension of the previous theory to higher-dimensional cases, whose mathematical background is presented in Section 4, together with some comments on the existence of second-order rigidity in Section 4.4. Section 5 contains illustrative sample problems showing low-level applicability of the simplex constraints, while the closing Section 6 gives the summary of the work done, completed by mentioning some further applications and problems that need investigation.

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2. Potential energy and tangent stiffness

If the expression of the total (internal and external) potential energy of an assembly is supplemented by some terms expressing the product of a kinematic *constraint function* with its Lagrange multiplier, the Hellinger–Reissner functional (Π_R) is obtained. According to the Hellinger–Reissner principle, the solution to this constrained extremum problem yields an equilibrium configuration of the assembly which also obeys the compatibility conditions. With the assumption of a linearly elastic material behaviour, it is possible to rewrite this functional in terms of displacement variables only, since Lagrange multipliers can then be regarded as internal force variables (Lanczos, 1986). Considering the formula (written, e.g. for bar-and-joint assemblies with m joints and n bars)

$$\Pi_R = -\sum_{\mu=1}^{dm} P_\mu\left(x_\mu - x_\mu^0\right) + \frac{1}{2}\sum_{\nu=1}^{n} g_\nu (e_\nu)^2 + \sum_{\nu=1}^{n} \widetilde{F}_\nu \Lambda_\nu, \qquad (1)$$

the first term gives the external potential (the negative of the work done by external forces): here d is the number of dimensions of the Euclidean space in which the assembly is analysed; P_μ stands for the μth external force component, while x_μ and x_μ^0 denote the μth current and reference coordinates, respectively. The next term describes the internal potential in terms of the *unloaded length* l_ν^u, axial stiffness g_ν (generally, $g_\nu = E_\nu A_\nu / l_\nu^u$) and elastic elongation e_ν of the νth member (bar). Note that 'unloaded length' in this context means the length of a member with no internal force *and* kinematic load (i.e., change in length due to thermal effect or manufacturing error). This is not to be confused with reference coordinates that can be assumed arbitrarily and therefore do not define necessarily the unloaded length of a bar member. Of course, if there exists a compatible configuration for an assembly with no kinematic loads or internal forces (that is not always the case), it is logical to consider the coordinates of one of such configurations as reference coordinates. Finally, in the third term of Π_R, Λ_ν is the Lagrange multiplier of the constraint function $\widetilde{F}_\nu$, whose canonical form is as follows:

$$\widetilde{F}_\nu = l_\nu(x_1, \ldots, x_{dm}) - l_\nu^u - e_\nu - t_\nu = 0. \qquad (2)$$

Here t_ν is the elongation component defined earlier as kinematic load. Let us call the attention to that t_ν is often used to denote *tension*, that is, axial force in the νth bar. In our case, axial force is denoted by Λ_ν. Writing the formula for linear elasticity,

$$e_\nu = \Lambda_\nu / g_\nu \qquad (3)$$

in both Eqs. (1) and (2), as well as introducing F_ν as

$$F_\nu = \widetilde{F}_\nu + e_\nu, \qquad (4)$$

Π_R can be transformed into the following form:

$$\Pi_R = -\sum_{\mu=1}^{dm} P_\mu\left(x_\mu - x_\mu^0\right) - \frac{1}{2}\sum_{\nu=1}^{n} \frac{\Lambda_\nu^2}{g_\nu} + \sum_{\nu=1}^{n} F_\nu \Lambda_\nu, \qquad (5)$$

dependent only on displacement and internal force variables. Consider now the first variation of Eq. (5):

$$\delta\Pi_R = \sum_{\mu=1}^{dm} \frac{\partial \Pi_R}{\partial x_\mu}\delta x_\mu + \sum_{\nu=1}^{n} \frac{\partial \Pi_R}{\partial \Lambda_\nu}\delta\Lambda_\nu. \qquad (6)$$

The system obeys equilibrium and compatibility conditions if and only if $\delta\Pi_R = 0$; thus, derivatives according to displacement variables yield the equilibrium equations as follows:

$$\sum_{\nu=1}^{n} \frac{\partial F_\nu}{\partial x_\mu}\Lambda_\nu - P_\mu = 0, \quad \mu = 1, \ldots, dm, \qquad (7)$$

or in matrix form:

$$\mathbf{A}\boldsymbol{\lambda} - \mathbf{p} = 0, \qquad (8)$$

where $\mathbf{A}$ is the equilibrium matrix and

$$\{A_{\mu\nu}\} = \frac{\partial F_\nu}{\partial x_\mu}, \quad \boldsymbol{\lambda}^{\mathrm{T}} = [\Lambda_1, \ldots, \Lambda_n], \quad \mathbf{p}^{\mathrm{T}} = [P_1, \ldots, P_{dm}].$$

Similarly, derivation according to internal force variables results in the system of compatibility equations

$$\widetilde{F}_\nu = 0, \quad \nu = 1, \ldots, n \qquad (9)$$

that can be linearized by considering constant and linear terms only in its Taylor expansion in the neighbourhood of reference coordinates as follows:

$$\sum_{\mu=1}^{dm} \frac{\partial F_\nu}{\partial x_\mu}\delta x_\mu + l_\nu^0 - l_\nu^u - e_\nu - t_\nu = 0, \quad \nu = 1, \ldots, n \qquad (10)$$

(l_ν^0 is the length of the νth bar when both of its connected nodes are at reference position). In matrix form:

$$\mathbf{Cd} - \mathbf{e}_e - \mathbf{e}_t = 0, \qquad (11)$$

where $\mathbf{C} = \mathbf{A}^{\mathrm{T}}$ and

$$\{C_{\nu\mu}\} = \frac{\partial F_\nu}{\partial x_\mu}, \quad \mathbf{d}^{\mathrm{T}} = [\delta x_1, \ldots, \delta x_{dm}],$$

$$\mathbf{e}_e^{\mathrm{T}} = [e_1, \ldots, e_n], \quad \mathbf{e}_t^{\mathrm{T}} = \left[-l_1^0 + l_1^u + t_1, \ldots, -l_n^0 + l_n^u + t_n\right].$$

Note that $l_\nu^u - l_\nu^0 = 0$ unless reference coordinates do not define the unloaded length for the νth bar.

After the equilibrium and compatibility equations having been derived, it will be shown how the state equation (i.e., that links nodal loads and displacements) of the displacement method follows from the Hellinger–Reissner principle. For this purpose, it is sufficient to assume only that compatibility conditions under Eq. (9) are satisfied: from Eq. (4) we obtain that $F_\nu = e_\nu$. Notice that it is not a strong condition, since arbitrary kinematic loads can be added to any member in order that the assembly can remain compatible with prescribed nodal positions. If Eq. (1) is rewritten in function of F_ν instead of Λ_ν, we have

$$\Pi_R^* = -\sum_{\mu=1}^{dm} P_\mu\left(x_\mu - x_\mu^0\right) + \frac{1}{2}\sum_{\nu=1}^{n} g_\nu F_\nu^2. \qquad (12)$$

Its first variation according to nodal displacements reads

$$\delta\Pi_R^* = \sum_{\mu=1}^{dm}\left(\sum_{\nu=1}^{n} g_\nu F_\nu \frac{\partial F_\nu}{\partial x_\mu} - P_\mu\right)\delta x_\mu, \qquad (13)$$

but can also be linearized using Eq. (10) as follows:

$$\delta\Pi_R^* = \sum_{\mu=1}^{dm}\left(\sum_{\nu=1}^{n} g_\nu\left(\sum_{\varphi=1}^{dm} \frac{\partial F_\nu}{\partial x_\varphi}\delta x_\varphi + l_\nu^0 - l_\nu^u - t_\nu\right)\frac{\partial F_\nu}{\partial x_\mu} - P_\mu\right)\delta x_\mu. \qquad (14)$$

The vanishing first variation means an equilibrium condition for the assembly; it can be written with matrix notation as

$$\mathbf{K}_a\mathbf{d} = \mathbf{q}, \qquad (15)$$

where $\mathbf{K}_a$ is the (linear or material) stiffness matrix, $\mathbf{q}$ is the vector of reduced loads and

$$\mathbf{K}_a = \mathbf{AGC}, \quad \mathbf{q} = \mathbf{p} - \mathbf{AGe}_t$$

with $\mathbf{G} = \langle g_1, \ldots, g_n\rangle$ being a diagonal matrix of axial stiffnesses. It is possible to derive a more general form of the state equation that also includes stiffness components (b) and (c), see the comments at Eq. (19)

The second variation of the potential energy over a compatible displacement field can be interpreted in slightly different senses. In the approach of Pellegrino and Calladine (1986) it provides information about the positivity of work done by internal and external forces on a displacement field $(\delta x_1,\ldots,\delta x_{dm})$. If Eq. (15) holds, the same variation gives answer for stability problems of that equilibrium configuration (Tarnai and Szabó, 2002). With or without obeying equilibrium conditions, however, it can also be regarded as the first variation of Eq. (13), i.e. a quadratic form with the tangent stiffness matrix:

$$\delta^2\Pi_R^* = \sum_{\varphi=1}^{dm}\sum_{\mu=1}^{dm}\left(\sum_{v=1}^{n} g_v \frac{\partial F_v}{\partial x_\varphi}\frac{\partial F_v}{\partial x_\mu} + \sum_{v=1}^{n} g_v F_v \frac{\partial^2 F_v}{\partial x_\mu \partial x_\varphi}\right)\delta x_\mu\,\delta x_\varphi. \tag{16}$$

With matrix notation, the first sum in the brackets corresponds again to $\mathbf{K}_a$, and therefore the second sum will be denoted by $\mathbf{K}_{bc}$ since it should represent both parts (b) and (c) of the tangent stiffness. Note that letters $\mathbf{G}$ for 'geometrical stiffness matrix' (Przemieniecki, 1968) and $\mathbf{H}$ for 'Hessian matrix' of $-\sum \Lambda_v F_v$ (Tarnai and Szabó, 2002) are also used for this latter component.

At the first sight, Eq. (16) still may seem to be insufficient to separate the effect of initial member force (b) and the geometrical stiffness (c). The key to this problem can be found in the *structure* of F_v as a function of nodal coordinates. This will be studied in details in the next section.

3. Possibilities of generalization

Looking back to the formulae of Section 2, they can be interpreted in a much wider context than just that of a bar-and-joint structure. It is possible to replace $\widetilde{F}_v$ by another constraint with different physical content (e.g. angle, area or volume, etc. instead of distance); and simultaneously, to introduce work-compatible Lagrange multipliers (moment, edge load or pressure, etc., respectively). This generalization preserves the validity of all former equations, provided e_v, t_v and g_v are understood as (general) elastic deformation, prescribed deformation and stiffness parameter (defined by Eq. (3)): it is easy to see that the potential function still exists due to the conservative nodal loads and the assumption of linear elasticity. There is only a restriction for the new constraint functions: first and second derivatives of F_v must exist at the examined configuration of the assembly.

In the forthcoming sections, only constraint functions defining a simplex in one, two or three dimensions will be dealt with. For the sake of simplicity, they will be called 'n-dimensional simplex constraints' (or n-simplex constraints) with $n = 1,2,3$ meaning length of a bar, area of a triangle and volume of a tetrahedron, respectively. If necessary, a right superscript (n) will indicate the current mechanical content, e.g. $l_v^{u(3)}$ means the stress-free volume of the vth tetrahedron of nodes. Note that if n is larger than one, these simplex constraints generally do not correspond to a unique shape for the simplex of $n+1$ nodes: configurations of the same area/volume can transform among each other through finite mechanisms.

Let us consider again the constraint function $\widetilde{F}_v$ of Eq. (2) together with F_v of Eq. (4), now in a general sense. Since all terms in F_v except for l_v are constant, it is sufficient to analyse only the current length (dimension). This is always a scalar–vector function, often written as a kind of norm which appears then in a form of a compound function:

$$l_v(x_1,\ldots,x_{dm}) = l_v(B_v(x_1,\ldots,x_{dm})). \tag{17}$$

With this, the second derivative of F_v can be written as follows:

$$\frac{\partial^2 F_v}{\partial x_\mu \partial x_\varphi} = \frac{d^2 F_v}{dB_v^2}\frac{\partial B_v}{\partial x_\mu}\frac{\partial B_v}{\partial x_\varphi} + \frac{dF_v}{dB_v}\frac{\partial^2 B_v}{\partial x_\mu \partial x_\varphi}. \tag{18}$$

Assuming also that $dF_v/dB_v \neq 0$, the second sum within the bracket of Eq. (16) can be rearranged (with $g_v F_v = \Lambda_v$), and thus for the tangent stiffness finally we have:

$$\delta^2\Pi_R^* = \sum_{\varphi=1}^{dm}\sum_{\mu=1}^{dm}\left(\sum_{v=1}^{n} g_v \frac{\partial F_v}{\partial x_\varphi}\frac{\partial F_v}{\partial x_\mu} - \sum_{v=1}^{n}\left[-\Lambda_v \frac{d^2F_v}{dB_v^2}\left(\frac{dF_v}{dB_v}\right)^{-2}\right]\frac{\partial F_v}{\partial x_\varphi}\frac{\partial F_v}{\partial x_\mu} + \sum_{v=1}^{n}\Lambda_v \frac{dF_v}{dB_v}\frac{\partial^2 B_v}{\partial x_\mu \partial x_\varphi}\right)\delta x_\mu\,\delta x_\varphi. \tag{19}$$

Three terms within the common brackets correspond here to $\mathbf{K}_a$, $\mathbf{K}_b$ and $\mathbf{K}_c$, respectively; and the expression in square brackets provides a general interpretation of which is called 'stress' in rigidity literature. If it is denoted by ω_v, the formal analogy with the concept of 'modified material stiffness' (Guest, 2006) becomes complete by writing together the first two sums in the bracket of Eq. (19) and $\mathbf{K}_b$. In a matrix form it reads:

$$\mathbf{K}_a + \mathbf{K}_b = \mathbf{A}(\mathbf{G} - \boldsymbol{\Omega})\mathbf{C}$$

with $\boldsymbol{\Omega} = \langle \omega_1,\ldots,\omega_n \rangle$ being a diagonal matrix of stress values. Note that the same terms $\mathbf{K}_a$, $\mathbf{K}_b$ and $\mathbf{K}_c$ would have been obtained in Eq. (15) if, instead of a pure linearization, second-order terms had also been taken into account in the Taylor expansion of F_v in Eq. (13). After plugging Eq. (18) into the expression of $\delta\Pi_R^*$, the more general formula $(\mathbf{K}_a + \mathbf{K}_b + \mathbf{K}_c)\mathbf{d} = \mathbf{q}$ of the state equation would have been derived.

Let us now compare the above statements with the classical bar-and-joint model. For the sake of a consistent notation, let the coordinates of the ith node N_i be given as a vector $\mathbf{x}_i = (x_{1,i}, x_{2,i}, x_{3,i})^{\mathrm{T}}$, and assume that the vth bar runs between N_1 and N_2. The constraint function of that bar is defined by the following relationships:

$$l_v^{(1)}\left(B_v^{(1)}\right) = \sqrt{B_v^{(1)}(x_{\alpha,i})}, \quad i = 1,2 \text{ and } \alpha = 1,2,3;$$

$$B_v^{(1)} = (x_{1,1} - x_{1,2})^2 + (x_{2,1} - x_{2,2})^2 + (x_{3,1} - x_{3,2})^2. \tag{20}$$

It is easy to verify that any first and second derivative of $B_v^{(1)}$,

$$\frac{\partial B_v^{(1)}}{\partial x_{\alpha,i}} = 2(x_{\alpha,i} - x_{\alpha,3-i}), \tag{21}$$

$$\frac{\partial^2 B_v^{(1)}}{\partial x_{\alpha,i}\partial x_{\beta,j}} = 2\delta_{\alpha\beta}(2\delta_{ij} - 1), \tag{22}$$

always exists (here δ_{ij} is the Kronecker symbol). Looking at the external derivatives,

$$\frac{dF_v^{(1)}}{dB_v^{(1)}} = \frac{1}{2\sqrt{B_v^{(1)}}}, \tag{23}$$

$$\frac{d^2F_v^{(1)}}{dB_v^{(1)2}} = -\frac{1}{4\sqrt{B_v^{(1)}}^3}, \tag{24}$$

they also always exist unless $B_v^{(1)} = 0$, which could only happen if N_1 and N_2 coincide. Noting also that Eq. (23) is never zero, it follows from (19) that

$$\omega_v^{(1)} = -\Lambda_v^{(1)}\left(-\frac{1}{4\sqrt{B_v^{(1)}}^3}\right)\left(\frac{1}{2\sqrt{B_v^{(1)}}}\right)^{-2} = \Lambda_v^{(1)}/\sqrt{B_v^{(1)}} = \Lambda_v^{(1)}/l_v^{(1)}, \tag{25}$$

in accordance with the original definition of 'stress'.

Although the third term within the brackets of (19), together with (22) and (23) yields a stress matrix with a general element $\Lambda_v^{(1)}/l_v^{(1)}\delta_{\alpha\beta}(2\delta_{ij} - 1)$ that seems to provide another general definition for $\omega_v^{(1)}$, this will not be the case for higher-order simplex constraints.

4. Simplex constraints

4.1. Mathematical formulation

Consider $n+1$ vectors $\mathbf{x}_i=(x_{1,i},x_{2,i},\dots,x_{d,i})^{\mathrm{T}}=\{x_{\alpha i}\}$, $\alpha=1,\dots,d$, $i=0,\dots,n$ (written in an orthogonal basis $\mathbf{i}_1,\dots,\mathbf{i}_d$ of a d-dimensional ($n\leqslant d$) vector space $\Re^d$) defining an n-simplex in $\Re^d$. Let us apply the notation $\mathbf{w}_i=\mathbf{x}_i-\mathbf{x}_0$, $i=1,\dots,n$ that allows the n-dimensional volume $l^{(n)}$ of such simplex to be written by the Gram determinant formula (Gritzmann and Klee, 1994) as follows:

$$l^{(n)}=\frac{1}{n!}\begin{vmatrix}\mathbf{w}_1^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_n\\ \vdots & \ddots & \vdots\\ \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_n\end{vmatrix}^{1/2}. \qquad (26)$$

Despite its compactness, the above formula has some disadvantages (e.g., the apparent asymmetry in vectors $\mathbf{x}_i$) from the aspect of further mathematical processes. Because of that, another formula will be used in order to compute $l^{(n)}$ in $\Re^d$. First of all, consider the expression

$$S^{(n)}_{j_1,\dots j_{d-n}}=\frac{1}{n!}\begin{vmatrix}1 & 1 & \dots & 1\\ \mathbf{x}_{0(j_1,\dots j_{d-n})} & \mathbf{x}_{1(j_1,\dots j_{d-n})} & \dots & \mathbf{x}_{n(j_1,\dots j_{d-n})}\end{vmatrix}, \qquad (27)$$

where the vectors are obtained by removing $d-n$ dimensions (deleting $d-n$ rows) of the original vectors $\mathbf{x}_i$ (lengthy superscripting lists all $d-n$ removed dimensions; the number of possible choices of $S^{(n)}_{j_1,\dots j_{d-n}}$ is then $\binom{d}{n}$). Note that $S^{(n)}_{j_1,\dots j_{d-n}}$ can be interpreted as the n-dimensional *signed* volume of the projection of the original n-simplex onto the n-dimensional subspace determined by non-removed dimensions. It will be shown that the formula

$$l^{(n)}=\left(\sum\left(S^{(n)}_{j_1,\dots j_{d-n}}\right)^2\right)^{1/2} \qquad (28)$$

(where the summation is made over $\binom{d}{n}$ n-subsets of d dimensions) is equivalent to Eq. (26).

Using the theorem that $|\mathbf{A}||\mathbf{A}|=|\mathbf{A}^{\mathrm{T}}\mathbf{A}|$ for a square matrix $\mathbf{A}$, Eq. (27) can be squared and rewritten in the form (instead of all entries, the determinant itself is subscripted once for clarity):

$$\left(S^{(n)}_{j_1,\dots j_{d-n}}\right)^2=\left(\frac{1}{n!}\right)^2\begin{vmatrix}1+\mathbf{x}_0^{\mathrm{T}}\mathbf{x}_0 & 1+\mathbf{x}_0^{\mathrm{T}}\mathbf{x}_1 & \dots & 1+\mathbf{x}_0^{\mathrm{T}}\mathbf{x}_n\\ 1+\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_0 & 1+\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_1 & \dots & 1+\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_n\\ \vdots & \vdots & \ddots & \vdots\\ 1+\mathbf{x}_n^{\mathrm{T}}\mathbf{x}_0 & 1+\mathbf{x}_n^{\mathrm{T}}\mathbf{x}_1 & \dots & 1+\mathbf{x}_n^{\mathrm{T}}\mathbf{x}_n\end{vmatrix}_{j_1,\dots j_{d-n}}. \qquad (29)$$

Now let us subtract the first row from all others, then the first column from the others: these operations do not affect the value of the determinant which can now be written in terms of $\mathbf{w}_i$ as follows:

$$\left(S^{(n)}_{j_1,\dots j_{d-n}}\right)^2=\left(\frac{1}{n!}\right)^2\begin{vmatrix}1+\mathbf{x}_0^{\mathrm{T}}\mathbf{x}_0 & \mathbf{x}_0^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{x}_0^{\mathrm{T}}\mathbf{w}_n\\ \mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0 & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_n\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{w}_n^{\mathrm{T}}\mathbf{x}_0 & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_n\end{vmatrix}_{j_1,\dots j_{d-n}}. \qquad (30)$$

The determinant itself can now be written as a sum of two determinants, that is:

$$\begin{vmatrix}1 & 0 & \dots & 0\\ 0 & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_n\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_n\end{vmatrix}_{j_1,\dots j_{d-n}}+\begin{vmatrix}\mathbf{x}_0^{\mathrm{T}}\mathbf{x}_0 & \mathbf{x}_0^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{x}_0^{\mathrm{T}}\mathbf{w}_n\\ \mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0 & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_n\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{w}_n^{\mathrm{T}}\mathbf{x}_0 & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_n\end{vmatrix}_{j_1,\dots j_{d-n}}. \qquad (31)$$

However, the second term will vanish, which can be seen from its dyadic product form below:

$$\left|\begin{bmatrix}\mathbf{x}_0^{\mathrm{T}}\\ \mathbf{w}_1^{\mathrm{T}}\\ \vdots\\ \mathbf{w}_n^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{x}_0 & \mathbf{w}_1 & \dots & \mathbf{w}_n\end{bmatrix}\right|_{j_1,\dots j_{d-n}}. \qquad (32)$$

Since any $\mathbf{x}_i$ here is n-dimensional, it is a minimal dyadic decomposition of an $n+1$-by-$n+1$ matrix with n dyads only, and its determinant is necessarily zero. Consequently,

$$\left(S^{(n)}_{j_1,\dots j_{d-n}}\right)^2=\left(\frac{1}{n!}\right)^2\begin{vmatrix}\mathbf{w}_1^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_n\\ \vdots & \ddots & \vdots\\ \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_1 & \dots & \mathbf{w}_n^{\mathrm{T}}\mathbf{w}_n\end{vmatrix}_{j_1,\dots j_{d-n}}. \qquad (33)$$

It is left only to see that the determinant in Eq. (26) is exactly the sum of all those seen in (33). Instead of a rigorous proof, we show rather give an example that can easily be generalized thereafter. Consider the determinant in Eq. (26) with $n=2$, $d=3$:

$$\begin{vmatrix}w_{11}w_{11}+w_{21}w_{21}+w_{31}w_{31} & w_{11}w_{12}+w_{21}w_{22}+w_{31}w_{32}\\ w_{12}w_{11}+w_{22}w_{21}+w_{32}w_{31} & w_{12}w_{12}+w_{22}w_{22}+w_{32}w_{32}\end{vmatrix}. \qquad (34)$$

Since a determinant is linear in its columns, this may be written as a sum of nine determinants as follows:

$$\begin{vmatrix}w_{11}w_{11} & w_{11}w_{12}\\ w_{12}w_{11} & w_{12}w_{12}\end{vmatrix}+\begin{vmatrix}w_{11}w_{11} & w_{21}w_{22}\\ w_{12}w_{11} & w_{22}w_{22}\end{vmatrix}+\dots+\begin{vmatrix}w_{31}w_{31} & w_{31}w_{32}\\ w_{32}w_{31} & w_{32}w_{32}\end{vmatrix}. \qquad (35)$$

In any of these nine second-order determinants, however, columns pertaining to the same dimension will be linearly dependent (e.g. the first determinant, whose first column multiplied by w_{12}/w_{11} equals the second one), hence only those assembled of columns pertaining to *all different* dimensions remain (e.g. the second determinant, whose first column is related to the first, the second column to the second dimension only, see the first subscript for the scalars w). Eventually, any ordered $n=2$-subset of column terms (pertaining to different dimensions) will appear exactly once as a determinant. Switching now to the other formulation, the sum of $\binom{3}{2}=3$ determinants in the fashion of Eq. (33) reads:

$$\begin{vmatrix}w_{11}w_{11}+w_{21}w_{21} & w_{11}w_{12}+w_{21}w_{22}\\ w_{12}w_{11}+w_{22}w_{21} & w_{12}w_{12}+w_{22}w_{22}\end{vmatrix}+\begin{vmatrix}w_{11}w_{11}+w_{31}w_{31} & w_{11}w_{12}+w_{31}w_{32}\\ w_{12}w_{11}+w_{32}w_{31} & w_{12}w_{12}+w_{32}w_{32}\end{vmatrix}+\begin{vmatrix}w_{21}w_{21}+w_{31}w_{31} & w_{21}w_{22}+w_{31}w_{32}\\ w_{22}w_{21}+w_{32}w_{31} & w_{22}w_{22}+w_{32}w_{32}\end{vmatrix}. \qquad (36)$$

Performing the same steps as before, it is obvious again from the linearity that only the determinants composed of columns pertaining to different dimensions will appear, exactly once. Consequently, the volume of a 2-simplex (i.e., area of a triangle) in 3D can be computed according to Eqs. (27) and (28) as follows:

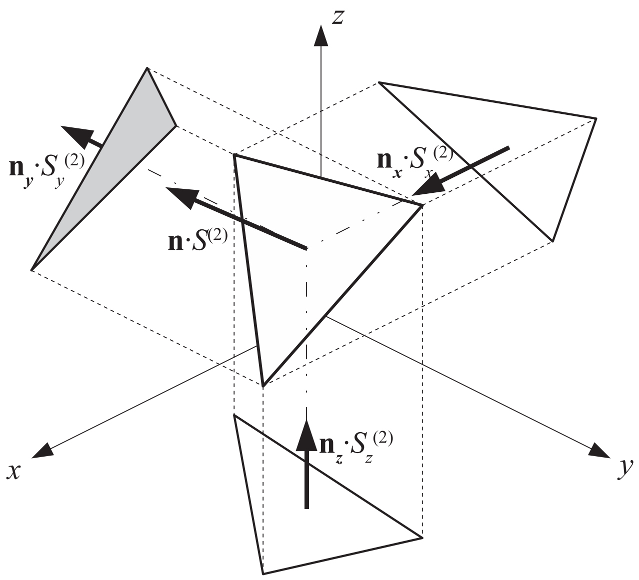

Fig. 1. Interpretation of the volume of a $d-1$-simplex in $\Re^d$ for the case $d = 3$. $S_\delta^{(2)}$ means here the (scalar and signed) volume of the projection of the original simplex onto the $d-1$-dimensional subspace orthogonal to the δth coordinate axis. Dark projection corresponds to negative signed volume; $\delta = x, y, z$, $\mathbf{n}$ and $\mathbf{n}_i$ are unit normal vectors.

$$l^{(2)} = \frac{1}{2!}\left(\begin{vmatrix} 1 & 1 & 1 \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 & 1 \\ x_{11} & x_{12} & x_{13} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 & 1 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}^2\right)^{1/2}. \tag{37}$$

A graphical interpretation of this result is shown in Fig. 1.

4.2. Cases of practical importance

In Euclidean spaces, of course, only six of the above listed possibilities (distance in one, two and three dimensions, area in two and three dimensions and volume in three dimensions) have any practical importance. As was seen in the preceding section, evaluation of first and second derivatives of determinants constitutes the basis for any numeric implementation. Let us consider first the formula, e.g. of a fourth-order determinant using tensor notation:

$$D^{(4)} = \sum_{\alpha,\beta,\gamma,\delta,i,j,k,l=1}^{4} x_{\alpha i}x_{\beta j}x_{\gamma k}x_{\delta l}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{ijkl},$$

where $\epsilon_{i\ j\ k\ l}$ is the Levi-Cività symbol giving +1 and -1 if i, j, k, l form an even or odd permutation of subscripts 1, 2, 3 and 4, respectively, and zero otherwise (note that although it is possible to give a general formulation for $D^{(n)}$, it does not help very much the understanding because of its multi-level subscripting). After replacement of an arbitrary dimension δ by a row of ones we obtain exactly $n!S_\delta^{(3)}$. Formulae for $S_\delta^{(3)}$ and its first and second derivatives are then as follows:

$$S_\delta^{(3)} = \frac{1}{3!}\sum_{i,j,k,l=1}^{4}\sum_{\alpha,\beta,\gamma=1}^{3} x_{\alpha i}x_{\beta j}x_{\gamma k}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{ijkl},$$
$$\frac{\partial S_\delta^{(3)}}{\partial x_{\alpha i}} = 3\cdot\frac{1}{3!}\sum_{j,k,l=1}^{4}\sum_{\beta,\gamma=1}^{3} x_{\beta j}x_{\gamma k}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{ijkl},$$
$$\frac{\partial^2 S_\delta^{(3)}}{\partial x_{\alpha i}\partial x_{\beta j}} = 2\cdot 3\cdot\frac{1}{3!}\sum_{k,l=1}^{4}\sum_{\gamma=1}^{3} x_{\gamma k}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{ijkl}.$$

This can easily be extended to smaller (higher) dimensions by subtracting (adding) +1 to each number and removing (inserting) a dimension in both subscripts. We note that, even if these summation formulae give a complete numerical foundation to any computation, it may be advisable to derive more compact forms (i.e., without summation) for particular cases to express directly the compound derivatives of F_v with respect to particular $x_{\alpha,i}$, $x_{\beta,j}$ coordinates. More details are presented in the Appendix.

4.3. Full-dimensional and degenerate constraints

Let a d-simplex in $\Re^d$ be called *full-dimensional*. Furthermore, let an n-simplex be called *degenerate* if $l^{(n)} = 0$. With reference to the formulae under Eqs. (17) and (27), a generalized simplex constraint function F_v reads

$$F_v^{(n)} = l_v^{(n)}(x_{\alpha,i}) - l_v^{u(n)} - t_v^{(n)} = 0, \quad \text{where}$$
$$l_v^{(n)}\left(B_v^{(n)}\right) = \sqrt{B_v^{(n)}(x_{\alpha,i})}, \quad i = 0,\dots,n \text{ and } \alpha = 1,\dots,d; \tag{38}$$
$$B_v^{(n)} = \sum\left(S_{j_1,\dots,j_{d-n}}^{(n)}(x_{\alpha,i})\right)^2,$$

see also Eq. (20) for the case of $n = 1$. Despite the uniform description, the constraint functions above are of two different kinds, depending on the dimension of the simplex.

(i) If a simplex constraint is not full-dimensional ($n < d$), formulae under (38) hold without change and derivatives of $l_v^{(n)}\left(B_v^{(n)}\right)$ can be computed as already written for the length constraint in 3D, see Eqs. (23) and (24):

$$\frac{dF_v^{(n)}}{dB_v^{(n)}} = \frac{1}{2\sqrt{B_v^{(n)}}}, \tag{39}$$

$$\frac{d^2F_v^{(n)}}{dB_v^{(n)2}} = -\frac{1}{4\sqrt{B_v^{(n)}}^3}. \tag{40}$$

Since these derivatives do not exist for $B_v^{(n)} = 0$, degenerate simplices *must not* be allowed (note that derivatives of internal functions still exist in any case), but otherwise the concept of stress can be extended according to Eq. (25) as follows:

$$\omega_v^{(n)} = \Lambda_v^{(n)}/l_v^{(n)}. \tag{41}$$

Relevant representatives of this class of simplex constraints are the common truss members (one-dimensional constraints) in either two or three dimensions, but triangular membrane elements (two-dimensional constraints) in three dimensions also belong here. It should be noted that such triangular constraints are suitable only for modelling shearless membranes like, e.g. the soap membrane, as the constraint function is sensible only to the area magnitude but not the shape of a triangle.

(ii-a) If a simplex constraint is full-dimensional ($n = d$) but degenerate configurations are still not allowed, formulae under (38), (23), (24) and (41) hold without change again, but $l_v^{(n)}$ can be computed simply as the absolute value of a determinant:

$$l_v^{(n)} = \text{abs}\left(\frac{1}{n!}\begin{vmatrix} 1 & 1 & \dots & 1 \\ \mathbf{x}_0 & \mathbf{x}_1 & \dots & \mathbf{x}_n \end{vmatrix}\right), \tag{42}$$

which is also equivalent to $\text{abs}\left(S_{n+1}^{(n)}(x_{\alpha,i})\right)$ in $d = n + 1$ dimensions. Two constraint types are important in this subclass: non-degenerate triangular and tetrahedral constraints in two and three dimensions, respectively (theoretically, there may be defined a signed length in a given direction to form a full-dimensional constraint in 1D, but it does not seem to have many applications). Tetrahedral constraints can be used for modelling 3D shearless continua like gases and fluids, while (co-)planar triangular ones seem to be applicable in 2D-models of the same materials. It is worth mentioning further applications in rigid kinematic modelling: for example, such a tetrahedral (triangular) constraint with three (two) fixed nodes constrain the remaining one to move finitely in a given plane (line), which is at least difficult to realize by pin-jointed truss members.

(ii-b) If a simplex constraint is full-dimensional ($n = d$), its constraint function can also be considered as the signed value of the determinant in Eq. (42) which now reduces to a dth-order polynomial, making possible to analyse the degenerate configurations, but stress (at least as suggested in Eq. (19)) cannot be defined there. Accordingly, the stiffness components $\mathbf{K}_b$ and $\mathbf{K}_c$ (i.e., the effect of initial internal forces) appear jointly as $\mathbf{K}_{bc}$, returning the Hessian matrix of $\Lambda_v \frac{1}{n!}\begin{vmatrix} 1 & 1 & \dots & 1 \\ \mathbf{x}_0 & \mathbf{x}_1 & \dots & \mathbf{x}_n \end{vmatrix} = \Lambda_v S^{(n)}_{n+1}(x_{\alpha,i})$ itself, see also the different structure of Eqs. (16) and (19) for explanation. This last subclass can be considered as the major innovation of generalized truss theory, since degenerate constraint in a common truss cannot be defined. Characteristically, they are applicable in modelling phenomena of lower dimension than the constraints are defined in: Example 1 in Section 5 will show the use of degenerate 2D simplices to simulate the behaviour of a *straight* scissor member, while in Example 2 degenerate tetrahedra will be used to model two straight elements moving in a common *plane*. With rigid material behaviour, this usage corresponds again to the kinematic modelling, but Example 2 also demonstrates how degenerate constraints are applicable in elastic models as well.

Note finally that *elastic* simplex constraints pertaining to any class above can freely be defined with zero unloaded length, area or volume (it is worth remarking here that soap membrane is a good example for an elastic assembly with practically zero unloaded area), since they are *not* said to be degenerate as far as they are analysed in a non-degenerate configuration: we recall that existence of the derivatives depends on the current n-dimensional length. In this sense, any elastic triangular (tetrahedral) simplex constraint with zero unloaded area (volume) can be considered as a type of generalization of 'zero free-length' springs discussed in Schenk et al. (2007).

4.4. Higher-order rigidity

Rigidity literature provides definitions for the so-called higher-order rigidity or stiffness; a brief overview and further references can be found in Kovács and Tarnai (2009). In our approach, a structure is considered to be rigid to the first order if and only if its compatibility matrix $\mathbf{C}$ has full rank, independently of a rigid or deformable material model ('first order' reflects that compatibility conditions form a linear equation system for the displacement components). If the nullspace of $\mathbf{C}$ is not empty, the structure is rather a first-order infinitesimal mechanism, but it still can have some stiffness against finite mobilization. Positivity of the second variation of Π^*_R is the condition of stable equilibrium, which can also hold if the contribution of stiffness members $\mathbf{K}_a$ and $\mathbf{K}_b$ vanish when a first-order mechanism is mobilised (we recall that a distinct $\mathbf{K}_b$ can only be defined in the presence of at least one non-degenerate simplex constraint). Let $\mathbf{D}$ represent a matrix containing all independent mechanisms column-wise in a certain basis, making possible to write any mechanism in the form $\mathbf{Da}$, where $\mathbf{a}$ is an arbitrary vector of coefficients. Now the second variation of Π^*_R reduces to the quadratic form $\mathbf{a}^{\mathrm{T}}\mathbf{W}\mathbf{a}$ with $\mathbf{W} = \mathbf{D}^{\mathrm{T}}\mathbf{K}_c\mathbf{D}$ called the *reduced form* (Guest, 2006; Tarnai and Szabó, 2002) of the tangent stiffness matrix. Clearly, sign definiteness of $\mathbf{W}$ is then a proof for the second-order rigidity (without respect to stiffness parameters again), since the geometrical stiffness member $\mathbf{K}_c$ is assembled of second derivatives. Higher-order rigidity conditions could then be analysed in the same way, using higher-order derivatives and tensorial objects.

Existence of such higher-order rigidity is crucial in common tensegrity trusses, since it is equivalent to the possibility of stabilizing of an equilibrium configuration by pre-stressing. Although numeric examples that can prove the following statement would exceed the limits of this paper, the simple analogy with common truss elements suggests that similar behaviour can be observed in structures with higher-dimensional non-degenerate simplex constraints. For example, confined triangular elements filling a bounded plane area seem to form an assembly that is rigid to the second order against any out-of-plane displacement, since it can be stabilized by a uniform tension as state of self-stress.

In the case of degenerate simplex constraints, $\mathbf{W}$ should be defined as $\mathbf{D}^{\mathrm{T}}\mathbf{K}_{bc}\mathbf{D}$ but all other statements about second-order rigidity still apply. Let us note an interesting feature of $\mathbf{K}_{bc}$ derived from any full-dimensional (but not necessarily degenerate) constraint, namely, that there are all zeros in its main diagonal due to the double derivation. Despite the first intuition, however, it does not exclude sign definiteness of $\mathbf{W}$, i.e. after the vector space of admissible displacement having been restricted (see Example 3).

5. Examples

5.1. Example 1: rigid bars with intermediate joints

In this example, a mechanical model of a structural solution (we call it 'non-breaking bar') is presented, where an extra node is introduced in the interior of a bar. It can be particularly useful in modelling scissor-like connections, used frequently in deployable structures (see e.g. You and Pellegrino, 1997). For this reason, let us consider a structural unit (shown in Fig. 2b) of the well-known 'Nuremberg scissors' (Fig. 2a).

Since our problem is two-dimensional, it can be assumed that the structure lies in the xy plane. The statical model departs from four bar (or 1D simplex) elements and is completed by two degenerate 2D simplex elements (that are necessarily full-dimensional) in order to prevent the scissor members from breaking at their midpoint. The equilibrium matrix of such a structure, with reference to Appendix B and writing c instead of $(a^2 + b^2)^{1/2}$, reads:

$$\mathbf{A} = \begin{array}{c|ccc|ccc|} & B2 & 21 & B21 & A2 & 23 & A23 \\ \hline x_1 & & a/c & -b/2 & & & \\ y_1 & & b/c & a/2 & & & \\ \hline x_2 & a/c & -a/c & b & a/c & -a/c & -b \\ y_2 & b/c & -b/c & -a & -b/c & b/c & -a \\ \hline x_3 & & & & & a/c & b/2 \\ y_3 & & & & & -b/c & a/2 \\ \hline \end{array}, \tag{43}$$

where the two- and three-character labels in the column headings refer to the corresponding length and area constraints, respectively. It can be shown that $\mathbf{A}$ has full rank, in addition, its inverse is

$$\mathbf{A}^{-1} = \frac{1}{2abc^2}\begin{bmatrix} 2b^3c & 2a^3c & bc^3 & ac^3 & 2bc^3 & 2ac^3 \\ 2a^2bc & 2ab^2c & & & & \\ -4ab^2 & 4a^2b & & & & \\ 2bc^3 & -2ac^3 & bc^3 & -ac^3 & 2b^3c & -2a^3c \\ & & & & 2a^2bc & -2ab^2c \\ & & & & 4ab^2 & 4a^2b \end{bmatrix}, \tag{44}$$

which makes possible to get the vector λ of internal forces from given nodal loads $\mathbf{p}$. If external nodes A and B are also taken into account by adding four rows to $\mathbf{A}$ according to the external reactions A_x, A_y, B_x and B_y, respectively (let this supplement be denoted by

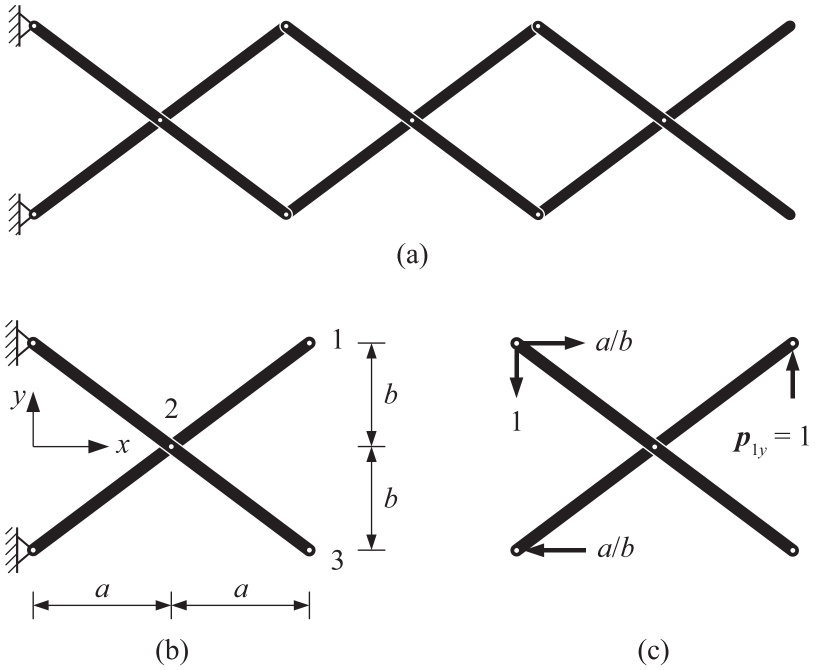

Fig. 2. Nuremberg scissors: the complete assembly (a); a structural unit composed of a pair of non-breaking bars represented by full-dimensional degenerate 2D simplex constraints (b); reaction forces due to a unit vertical load applied at node 1 (c).

$\mathbf{A}_{ext}$), the 4-element vector $\mathbf{p}_{ext}$ of reactions at A and B can be computed as follows:

$$\mathbf{p}_{ext} = \mathbf{A}_{ext}\mathbf{A}^{-1}\mathbf{p}. \tag{45}$$

Considering now the elements of the coefficient matrix,

$\mathbf{A}_{ext}\mathbf{A}^{-1} =$		p_{1x}	p_{1y}	p_{2x}	p_{2y}	p_{3x}	p_{3y}
	A_x	-1	a/b	$-1/2$	$a/(2b)$		a/b
	A_y	b/a	-1	$b/(2a)$	$-1/2$	b/a	
	B_x		$-a/b$	$-1/2$	$-a/(2b)$	-1	$-a/b$
	B_y	$-b/a$		$-b/(2a)$	$-1/2$	$-b/a$	-1

(46)

reaction forces due to elementary loading cases can be read column-wise. For instance, two columns in the middle reflect a purely truss-like behaviour, while a vertical load applied at node 1 results in reactions drawn in Fig. 2c. (Note just as an illustration that by replicating such scissor-like structural units, one can obtain a $2an$-long structure, where the external reactions from forces $\mathbf{p}_i$ ($i \leqslant n$) applied in the middle or at the right hand side of the ith scissor element can be obtained from the matrix power expression $-[(-\mathbf{A}_{ext}\mathbf{A}^{-1})^*]^i\mathbf{p}_i$, where $(-\mathbf{A}_{ext}\mathbf{A}^{-1})^*$ is generated by introducing two empty rows into the middle of $\mathbf{A}_{ext}\mathbf{A}^{-1}$.)

5.2. Example 2: sliding elastic bars

Tarnai and Makai (1989) investigated a finite mechanism composed of two tetrahedral trusses, where each edge (bar) of one tetrahedron was constrained to slide along a bar of the other tetrahedron. By neglection of the thickness of the bars, each pair of connected bars can be regarded as a degenerate 3D simplex constraint. Although the referenced mobility analysis of tetrahedra was based on a kinematic constraint function like Eq. (2), only the kinematic behaviour of a rigid assembly has been analysed. In our example, however, a complex static–kinematic approach of such sliding pair of bars with elastic material behaviour will be presented.

Consider two connected bars of length l_1 and l_2 running between points A, C and B, D, respectively, as shown in Fig. 3. Let

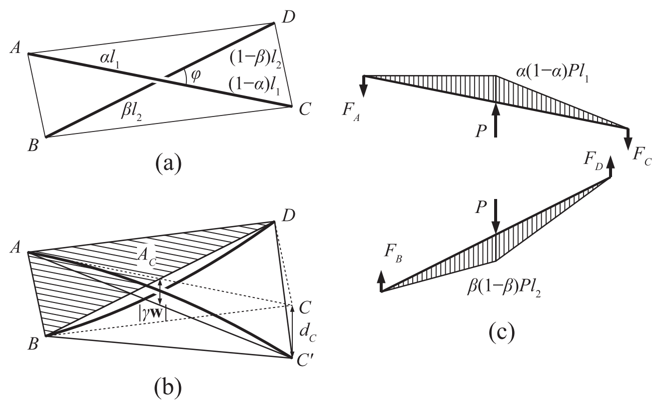

Fig. 3. Pair of sliding bars: dimensions of an unloaded configuration, i.e. where the bar axes intersect (a), deformed shape of connected elastic bars: the distance $|\gamma\mathbf{w}|$ between lines AC' and BD (obtained here from the modified position vector $\mathbf{c}'$) can be considered parallel to d_C because of the assumption of small displacements (b) and internal moment diagrams (drawn onto the tensile side) of connected bars (c).

the position vectors of A, B, C, D be denoted by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, respectively. Furthermore, let a vector pointing along the normal transversal of the bars be defined as $\mathbf{w} = (\mathbf{c}-\mathbf{a}) \times (\mathbf{d}-\mathbf{b})$. Writing the position vector of the intersection of the normal transversal and BD in two different ways we have:

$$\mathbf{a} + \alpha(\mathbf{c}-\mathbf{a}) + \gamma\mathbf{w} = \mathbf{b} + \beta(\mathbf{d}-\mathbf{b}), \tag{47}$$

where α, β and γ are scalars and any of them can be determined easily by forming scalar triple products with the (vectorial) coefficients of the two other scalars. The formal results are as follows:

$$\alpha = \frac{((\mathbf{c}-\mathbf{a}) \times (\mathbf{d}-\mathbf{b}))^{\mathrm{T}}((\mathbf{b}-\mathbf{a}) \times (\mathbf{d}-\mathbf{b}))}{((\mathbf{c}-\mathbf{a}) \times (\mathbf{d}-\mathbf{b}))^2}, \tag{48}$$

$$\beta = \frac{((\mathbf{c}-\mathbf{a}) \times (\mathbf{d}-\mathbf{b}))^{\mathrm{T}}((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}))}{((\mathbf{c}-\mathbf{a}) \times (\mathbf{d}-\mathbf{b}))^2}, \tag{49}$$

$$\gamma = \frac{(\mathbf{b}-\mathbf{a})(\mathbf{c}-\mathbf{a})(\mathbf{d}-\mathbf{b})}{((\mathbf{c}-\mathbf{a}) \times (\mathbf{d}-\mathbf{b}))^2}. \tag{50}$$

The last expression reflects that intersecting (straight) bars make γ vanish. From now on, suppose that α and β are known in a given compatible configuration of *ABCD* that defines a 3D simplex constraint. If the assembly becomes loaded, a force P appears in a direction perpendicular both to *AC* and *BD* as shown in Fig. 3c and, if the bending stiffness of the bars EI_1 and EI_2 is finite, it comes together with a deformation sketched in Fig. 3b. In order to simplify further calculations of stiffness parameter $g^{(3)}$, we take the assumption of small displacements. In this case, the deformed shape of the (originally degenerate) tetrahedron *ABCD* can be characterized by a single parameter: by keeping nodes A, B and D fixed, that parameter will be the off-plane displacement d_C of node C. Using, e.g. the principle of virtual work, d_C can be expressed (with respect to $F_C = \alpha P$ from the condition of equilibrium) as follows:

$$d_C = \frac{P}{3\alpha}\left(\frac{l_1^3}{EI_1}\alpha^2(1-\alpha)^2 + \frac{l_2^3}{EI_2}\beta^2(1-\beta)^2\right). \tag{51}$$

Since the internal work of the elastic deformation can also be written in terms of pressure and volumetric increase, the following relationship is obtained:

$$F_C d_C = p\Delta V, \tag{52}$$

where $\Delta V = A_C d_C/3$, and $A_C = l_1 l_2 \alpha \sin\varphi/2$, using the notation of Fig. 3a and b. Plugging these latter expressions into Eq. (52), after rearrangement we have

$$p = \frac{6P}{l_1 l_2 \sin\varphi}. \tag{53}$$

We recall that here $p = \Lambda^{(3)}$ and $\Delta V = e^{(3)}$ with our general notation. Thus, with Eq. (3) the stiffness parameter $g^{(3)}$ can be expressed as follows:

$$\begin{aligned} g^{(3)} &= p\Delta V^{-1} = \frac{6P}{l_1 l_2 \sin\varphi}\,\frac{6}{l_1 l_2 \alpha \sin\varphi d_C} \\ &= \frac{108}{(l_1 l_2 \sin\varphi)^2}\left(\frac{l_1^3}{EI_1}\alpha^2(1-\alpha)^2 + \frac{l_2^3}{EI_2}\beta^2(1-\beta)^2\right)^{-1}. \end{aligned} \tag{54}$$

We note that this stiffness formula with coefficients derived from Eqs. (48)–(50) also holds if *ABCD* is not coplanar in a given configuration, provided the displacements are still small enough to accept the relationship of Eq. (51).

After such a preliminary study on stiffnesses, let us consider an assembly composed of two degenerate 3D simplex constraints according to Fig. 4 as a statical model of a primitive bridge made of overlapping pin-jointed beams. We want to determine the reactions and deflections due to the force Q.

Assume that four bars have infinite normal stiffness but their bending stiffness EI is finite. Let all external (supported) nodes be labelled by letters A, B, C, D to distinguish internal (unsupported) nodes that are numbered. Because of the inextensional mode, the assembly has only two kinematical degrees of freedom (displacements along z_1 and z_2), hence the four first-order simplex constraints represented by the bars can be left out of consideration in the following. In accordance with paragraph (ii-b) in Section 4.3, let the four nodes be arranged in the constraint functions in counterclockwise order 12*AB* and 21*CD*, giving

$$l_1^{(3)} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ c & b & 0 & 0 \\ 0 & a & a & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad l_2^{(3)} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & c & b+c & b+c \\ a & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{vmatrix}. \tag{55}$$

The nonzero derivatives (with respect to z_1 and z_2) are then

$$\frac{\partial F_1^{(3)}}{\partial z_1} = -\frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ b & 0 & 0 \\ a & a & 0 \end{vmatrix} = -\frac{ab}{6}, \quad \frac{\partial F_1^{(3)}}{\partial z_2} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ c & 0 & 0 \\ 0 & a & 0 \end{vmatrix} = \frac{ac}{6},$$

$$\frac{\partial F_2^{(3)}}{\partial z_1} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ b & b+c & b+c \\ a & 0 & a \end{vmatrix} = \frac{ac}{6},$$

$$\frac{\partial F_2^{(3)}}{\partial z_2} = -\frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ c & b+c & b+c \\ 0 & 0 & a \end{vmatrix} = -\frac{ab}{6}.$$

Note that each of the above elements (arranged anyway as in the compatibility matrix) corresponds to one third of the triangle area $A_{\nu,i}$ opposite to node i in the νth constraint. The first important observation is that $\mathbf{C}$ is nonsingular if $b \neq c$, showing that our assembly has no first-order mobility. From the equilibrium Eq. (8), all (generalized) internal forces can be determined explicitly:

$$\boldsymbol{\lambda} = \mathbf{C}^{-\mathrm{T}}\mathbf{p} = \frac{6}{a(c^2-b^2)}\begin{bmatrix} b & c \\ c & b \end{bmatrix}\begin{bmatrix} -Q \\ 0 \end{bmatrix} = \frac{6Q}{a(b^2-c^2)}\begin{bmatrix} b \\ c \end{bmatrix}. \tag{56}$$

External reactions can now be obtained using the full equilibrium matrix, completed by four rows corresponding to supported nodes in the order A, B, C, D at the bottom as follows:

$$\mathbf{p}_{full} = \mathbf{C}_{full}^{\mathrm{T}}\boldsymbol{\lambda} = \frac{a}{6}\begin{bmatrix} -b & c \\ c & -b \\ -c & \\ b & \\ & -c \\ & b \end{bmatrix}\begin{bmatrix} b \\ c \end{bmatrix}\frac{6Q}{a(b^2-c^2)} = \frac{Q}{b^2-c^2}\begin{bmatrix} 0 \\ 0 \\ -bc \\ b^2 \\ -c^2 \\ bc \end{bmatrix}. \tag{57}$$

The results above are shown in Fig. 5a. Deflections are calculated using the stiffness matrix. The 2-by-2 matrix $\mathbf{C}$ is used again to evaluate $\mathbf{K}_a = \mathbf{C}^{\mathrm{T}}\mathbf{G}\mathbf{C}$, giving:

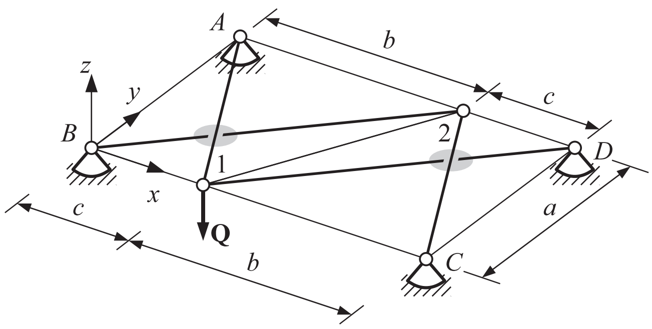

Fig. 4. Structure with two 3D degenerate simplex constraints: empty circles denote ball joints, gray patches symbolize sliding connections. Only the bending stiffness of the bars is considered to be finite.

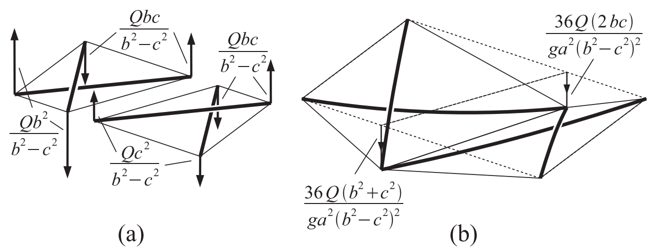

Fig. 5. Reactions (a) and deflections (b) of the example structure. Note the different curvatures of the deflected beams: in a physical model, members *A*1 and 2*C* would run above their connected counterpart.

$$\mathbf{K}_a = \frac{a}{6}\begin{bmatrix} -b & c \\ c & -b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix} \begin{bmatrix} -b & c \\ c & -b \end{bmatrix} \frac{a}{6} = \frac{a^2}{36}\begin{bmatrix} g_1 b^2 + g_2 c^2 & -(g_1+g_2)bc \\ -(g_1+g_2)bc & g_1 b^2 + g_2 c^2 \end{bmatrix}. \quad (58)$$

We note that $\mathbf{K}_{bc}$ vanishes, since the displacements of nodes 1 and 2 are both vertical, and any second derivative of a determinant $F_i^{(3)}$ with respect to z coordinates is zero. With a further assumption that $g_1 = g_2 = g$ (superscript (3) is omitted for brevity) we have

$$\mathbf{K} = \frac{ga^2}{36}\begin{bmatrix} b^2+c^2 & -2bc \\ -2bc & b^2+c^2 \end{bmatrix}. \quad (59)$$

Finally, the two-element deflection vector is obtained by solving the equation of state $\mathbf{Kd} = \mathbf{p}$ as follows:

$$\mathbf{d} = \frac{36}{ga^2(b^2-c^2)^2}\begin{bmatrix} b^2+c^2 & 2bc \\ 2bc & b^2+c^2 \end{bmatrix}\begin{bmatrix} -Q \\ 0 \end{bmatrix} = \frac{36Q}{ga^2(b^2-c^2)^2}\begin{bmatrix} b^2+c^2 \\ 2bc \end{bmatrix}, \quad (60)$$

see also the graphics in Fig. 5b.

5.3. Example 3: 3D simplex constraint and second-order rigidity

Consider the previous assembly with a simplified geometry ($c = b$). Instead of diagonal bracing elements, let nodes 1 and 2 be constrained by two sliders of angle α and β to the horizontal in planes perpendicular to y as shown in Fig. 6 (the two degenerate simplex constraints remain: both can now be imagined as a compound of two connected telescopic bars, which are not drawn in the figure). Considering nodes 1 and 2 being constrained by a slider of a single kinematical degree of freedom each, matrix $\mathbf{C}$ can now be written, e.g. as

$$\mathbf{C} = \begin{bmatrix} 0 & -ac/6 & 0 & ac/6 \\ 0 & ac/6 & 0 & -ac/6 \\ \sin\alpha & -\cos\alpha & 0 & 0 \\ 0 & 0 & \sin\beta & -\cos\beta \end{bmatrix}, \quad (61)$$

where the order of columns and rows correspond to x_1, z_1, x_2, z_2, as well as $12AB$, $21CD$, slider 1, slider 2, respectively. It follows from a minimal dyadic decomposition of $\mathbf{C}$ that its rank deficiency is 1, so there is only one state of self-stress λ and one infinitesimal mechanism $\mathbf{d}$ for the assembly. It can be shown by a simple left multiplication by $\mathbf{C}$ or $\mathbf{C}^{\mathrm{T}}$ that $\lambda^{\mathrm{T}} = [\Lambda\ \Lambda\ 0\ 0]$; $\mathbf{d}^{\mathrm{T}} = [\cot\alpha\ 1\ \cot\beta\ 1]$. Let us check whether or not the assembly can be stiffened by pre-stressing: since the volume constraints are degenerate, $\mathbf{K}_{bc}$ should be assembled. For example, the top right element of that 4-by-4 matrix is obtained from the following scheme:

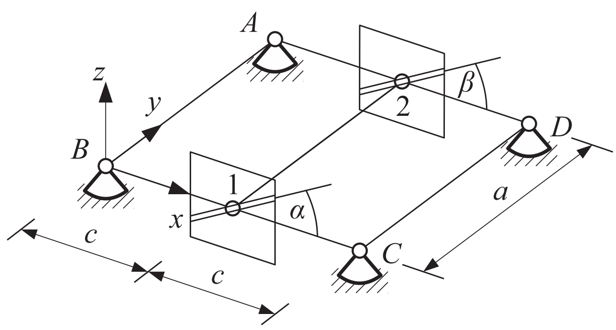

Fig. 6. Assembly composed of two degenerate tetrahedral constraints. Double lines represent oblique sliders with a single kinematical degree of freedom.

$$\Lambda_1\frac{\partial^2 F_1^{(3)}}{\partial x_1\,\partial z_2} + \Lambda_2\frac{\partial^2 F_2^{(3)}}{\partial x_1\,\partial z_2} = \frac{\Lambda}{6}\left((-1)(+1)\begin{vmatrix} 1 & 1 \\ a & 0 \end{vmatrix} + (+1)(+1)\begin{vmatrix} 1 & 1 \\ 0 & a \end{vmatrix}\right), \quad (62)$$

where bracketed coefficients refer to the successive use of checkerboard rule in determinant expansion. With the complete matrix,

$$\mathbf{K}_{bc} = \frac{\Lambda a}{3}\begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}, \quad (63)$$

the reduced form of the complementary stiffness matrix (now of size 1-by-1) can be obtained as follows:

$$\mathbf{D} = \mathbf{d}^{\mathrm{T}}\mathbf{K}_{bc}\mathbf{d} = \frac{2\Lambda a}{3}(\cot\alpha - \cot\beta). \quad (64)$$

This result shows that once $\beta \neq \alpha$, $\mathbf{D}$ is sign definite, therefore our assembly, even if an infinitesimal mechanism, is rigid to the second order.

6. Discussion

In the present paper, an attempt was made to describe the relationship of different kinematic constraints and stiffness in a unified framework on the basis of a potential energy principle. It was proved that the common truss theory is based on a special kinematic constraint which is part of a broader set, called the set of 'n-dimensional simplex constraints'. After the introduction of the concept of 'full-dimensional simplex constraint', it has also been shown that such constraints can be formulated with zero volume as well, making possible to insert earlier mathematical models (e.g. for sliding connected bars) into the global truss theory generalized by simplex constraints. In addition, the concept of 'stress' used in the literature has been extended to the set of simplex constraints. The presented way of derivation of stress implies its existence if the constraint function is a compound function expressing metric properties, therefore no stress value can be associated to degenerate constraints. Three examples were attached to illustrate the application of simplex constraints assuming rigid as well as elastic material behaviour, completed by investigations on second-order rigidity which led to the statement that even zero-volume (degenerate) constraints can result in additional (second-order) stiffness via pre-stressing.

A question can still arise about why non-degenerate 2D and 3D constraints are missing from those examples. The obvious answer is that such constraint types can be used to advantage in kinematically indeterminate systems like some models for membrane structures and shearless continua, where the equilibrium configuration is to be found. Although iterative form-finding procedures used in cable networks could easily be extended to simplex constraints as well, the computational work of such problems exceeds the limits of brief parametric examples.

In the lack of such experiences, further investigations may be necessary to clarify the applicability of higher-dimensional constraints in analysing bubble-like structures. In this aspect, an extra consideration can also be added by mentioning that a constraint function formulated as a sum of other constraint functions rules an 'average' behaviour characterized by 'unified' variables (for clarity: if the sum of the length of all sides of a triangle is constrained, the corresponding force parameter can be understood as a (unique) force arising in a rope bounding the triangle), which probably allows the simultaneous control, e.g. over the volume and surface parameters of bubbles in foams.

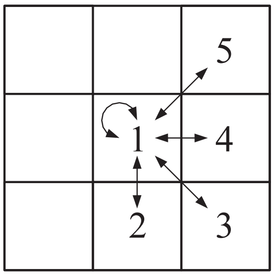

Fig. A.1. Possible relative positions of any two of the nine coordinates $x_{\alpha,i}$. The coordinates can always be permuted such that a given coordinate is moved to the center (x_{22}) without changing the value of any determinant S_δ ($\delta = x, y, z$).

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Appendix A. Area constraints in three dimensions

Let an area constraint in the 3D space be defined by three non-collinear nodes $\mathbf{x}_i = (x_{1,i}, x_{2,i}, x_{3,i})^{\mathrm{T}}$ ($i = 1, 2, 3$) bounding an area $l_v^{(2)}$ (from now on, subscripts v are omitted for clarity). Consider Eq. (37) as the point of departure. It can be rewritten with the formalism of Eq. (38) as

$$l^{(2)} = (B^{(2)})^{1/2},$$
$$B^{(2)} = (S_3^{(2)})^2 + \left(S_2^{(2)}\right)^2 + \left(S_1^{(2)}\right)^2. \tag{A.1}$$

If both subscripts i and α are interpreted cyclically as $1, 2, 3, 1, \ldots$ henceforth, we can write for the first derivative of $B^{(2)}$ that

$$\frac{\partial B^{(2)}}{\partial x_{\alpha,i}} = 2\left(S_{\alpha+1}^{(2)} \frac{\partial S_{\alpha+1}^{(2)}}{\partial x_{\alpha,i}} + S_{\alpha+2}^{(2)} \frac{\partial S_{\alpha+2}^{(2)}}{\partial x_{\alpha,i}}\right)$$
$$= 2\left(S_{\alpha+1}^{(2)}(x_{\alpha+2,i+2} - x_{\alpha+2,i+1}) - S_{\alpha+2}^{(2)}(x_{\alpha+1,i+2} - x_{\alpha+1,i+1})\right). \tag{A.2}$$

In order to write the second derivatives as well, two extra considerations will be needed. We recall first that any odd-order determinant is invariant under a permutation of rows (columns) that preserves their (cyclic) order: it makes possible to rearrange the 3-by-3 matrix of coordinates such that $x_{\alpha,i}$ is the central element; consequently, the possible kinds of second derivatives are reduced to nine. Secondly, the sequence of derivations is reversible, so there exist only five relative positions for any two of the coordinates as shown in Fig. A.1.

With the above considerations, the five kinds of second derivatives are as follows (written according to the figure for the sake of compactness, but general formulae in the fashion of Eq. (A.2) can be obtained by adding $\alpha - 2$ and $i - 2$ in a cyclic sense to the first and second subscripts, respectively):

$$\frac{\partial^2 B^{(2)}}{\partial x_{22}^2} = 2(x_{13} - x_{11})^2 + 2(x_{33} - x_{31})^2,$$
$$\frac{\partial^2 B^{(2)}}{\partial x_{22}\,\partial x_{32}} = 2(x_{21} - x_{23})(x_{33} - x_{31}),$$
$$\frac{\partial^2 B^{(2)}}{\partial x_{22}\,\partial x_{33}} = 2(x_{22} - x_{21})(x_{33} - x_{31}) + 2S_1^{(2)}, \tag{A.3}$$
$$\frac{\partial^2 B^{(2)}}{\partial x_{22}\,\partial x_{23}} = 2(x_{12} - x_{11})(x_{11} - x_{13}) + 2(x_{31} - x_{32})(x_{33} - x_{31}),$$
$$\frac{\partial^2 B^{(2)}}{\partial x_{22}\,\partial x_{13}} = 2(x_{21} - x_{22})(x_{11} - x_{13}) - 2S_3^{(2)}$$

Completed by formulae under Eqs. (39) and (40), everything is ready to be plugged into either Eqs. (7), (10) or (18). The definition formula for the generalized stress, similar to (25), reads:

$$\omega^{(2)} = \Lambda^{(2)}/l^{(2)}, \tag{A.4}$$

but $\Lambda^{(2)}$ is now a force distributed over a distance; therefore, $\omega^{(2)}$ (like the stiffness parameter $g^{(2)}$) has a unit of force over distance to the third power.

Appendix B. Full-dimensional area constraints

Let an area constraint be defined by three coplanar nodes $\mathbf{x}_i = (x_{1,i}, x_{2,i})^{\mathrm{T}}$ ($i = 1, 2, 3$) bounding an area $l^{(2)}$ in a 2D vector space. In contrast to Eq. (A.1), $B^{(2)}$ should either be written as single-term expression $\sqrt{\left(S_3^{(2)}\right)^2} = \mathrm{abs}\left(S_3^{(2)}\right)$ or, if a degenerate configuration is analysed, simply as $S_3^{(2)}$. Note that in this latter case, the sign of signed area $S_3^{(2)}$ is determined by the succession of nodes within the constraint function. First and second derivatives of $S_3^{(2)}$ with respect to $x_{\alpha,i}$ can be computed with cyclic subscripts $\alpha = 1, 2, 1, \ldots$, $i = 1, 2, 3, 1, \ldots$ directly as follows:

$$\frac{\partial S_3^{(2)}}{\partial x_{\alpha,i}} = \frac{1}{2}(x_{\alpha+1,i+2} - x_{\alpha+1,i+1})(-1)^\alpha, \tag{B.1}$$
$$\frac{\partial^2 S_3^{(2)}}{\partial x_{\alpha,i}\,\partial x_{\alpha+1,i+1}} = -\frac{1}{2}(-1)^\alpha \quad \text{and zero otherwise}. \tag{B.2}$$

Note that the above expression for the first derivative have already been referred in Eq. (A.2). As an immediate consequence of Eq. (B.2), the matrix $\mathbf{K}_{bc}$ of any individual full-dimensional area constraint has the following form:

$$\mathbf{K}_{bc} = \begin{array}{c|c|c|c|c|c|c|} & x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \\ \hline x_1 & & & & 1 & & -1 \\ y_1 & & & -1 & & 1 & \\ x_2 & & -1 & & & & 1 \\ y_2 & 1 & & & & -1 & \\ x_3 & & 1 & & -1 & & \\ y_3 & -1 & & 1 & & & \\ \hline \end{array} \;\frac{1}{2}\Lambda^{(2)}. \tag{B.3}$$

Appendix C. Volume constraints

Consider finally a volume constraint (which is necessarily full-dimensional in 3D) defined by four nodes $\mathbf{x}_i = (x_{1,i}, x_{2,i}, x_{3,i})^{\mathrm{T}}$ ($i = 1, 2, 3, 4$) bounding a volume $l^{(3)}$. Let us choose the interpretation $l^{(3)} = S_4^{(3)}$ (i.e., degenerate constraints are allowed). Due to the increased number of variables, the derivative formulae will be given here, instead of cyclic subscripting, with the aid of subdeterminants $D_{(\alpha,\beta,\ldots)(i,j,\ldots)}$ meaning determinants obtained by deletion of row(s) $\alpha, \beta, \ldots$ and column(s) $i, j, \ldots$ of $S_4^{(3)}$. For the first and second derivatives we have then

$$\frac{\partial S_4^{(3)}}{\partial x_{\alpha,i}} = \frac{1}{6}(-1)^{\alpha+i} D_{(\alpha)(i)}, \tag{C.1}$$
$$\frac{\partial^2 S_4^{(3)}}{\partial x_{\alpha,i}\,\partial x_{\beta,j}} = \frac{1}{6}(-1)^{\alpha+\beta+i+j}\,\mathrm{sgn}((\alpha-\beta)(i-j))\,D_{(\alpha,\beta)(i,j)}. \tag{C.2}$$

If degenerate cases are excluded, it is still possible to choose the interpretation $l^{(3)} = \sqrt{\left(S_3^{(2)}\right)^2}$. This modification implies two consequences:

(i) in both derivative expressions, the results should be multiplied with the sign of $S_3^{(2)}$;
(ii) the generalized stress,

$$\omega^{(3)} = \Lambda^{(3)}/l^{(3)} \quad \text{(C.3)}$$

also becomes possible to interpret similar to that shown at the non-degenerate area constraints.

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